Design of a Ship's Propeller Blade

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Mathematics Higher Level IB

Type III Assignment

A ship's propeller blade is to be modelled by three functions, $P(x)$, $Q(x)$ and $R(x)$. For each of these functions one unit represents 10cm and x is the horizontal distance along the drive shaft to which that height will be attached. Restrictions are placed on the functions such that $2 \leq P(x) \leq 3$, $3 \leq Q(x) \leq 12$ and $12 \leq R(x) \leq 21$. It has been found that $P(x)$ and $Q(x)$ are cubic and $R(x)$ is linear. The functions are related by six equations:

$$\begin{aligned}
P(2) &= 3 && (1)\\
P(3) &= 7 \\
Q(3) &= 7 && (2)\\
Q(12) &= 12 && (3)\\
P'(3) &= Q'(3) && (4)\\
P''(3) &= Q''(3) && (5)\\
P''(2) &= 0 \\
Q''(12) &= 0 && (6)
\end{aligned}$$

From (1) ensures that the curved part of the blade begins at the correct location. (2) guarentees that the two parts meet, (4) and (5) ensure the smoothness of the joining of $P(x)$ and $Q(x)$. (6) ensures that the ends of each part are smooth.

From the dominant terms of the equations given above we can perform the following definitions:

$$\begin{aligned}
\text{Let} \quad P(x) &= p_3x^3 + p_2x^2 + p_1x + p_0 \\
\text{Let} \quad Q(x) &= q_3x^3 + q_2x^2 + q_1x + q_0 \\
\text{Let} \quad R(x) &= r_1x + r_0
\end{aligned}$$

1 Simultaneous Equations

From the information given above we can derrive a system of simultaneous equations.

$$\begin{aligned}
8p_3 + 4p_2 + 2p_1 + p_0 &= 3 && (7)\\
27p_3 + 9p_2 + 3p_1 + p_0 &= 7 \\
&= 27q_3 + 9q_2 + 3q_1 + q_0 && (8)\\
1728q_3 + 144q_2 + 12q_1 + q_0 &= 12 && (9)\\
27p_3 + 6p_2 + p_1 &= 27q_3 + 6q_2 + q_1 && (10)\\
18p_3 + 2p_2 &= 18q_3 + 2q_2 && (11)\\
12p_3 + 2p_2 &= 0 \\
&= 72q_3 + 2q_2 && (12)
\end{aligned}$$

Some relations are used:

$$\frac{\mathrm{d}P(x)}{\mathrm{d}x} = P'(x) = 3p_3x^2 + 2p_2x + p_1$$
$$\frac{\mathrm{d}^2P(x)}{\mathrm{d}x^2} = P''(x) = 6p_3x + 2p_2$$

and $Q'(x)$, $Q''(x)$ similarly.

$$(11) - (12) \Rightarrow 6p_3 = -54q_3$$
$$\Rightarrow p_3 = -9q_3 \qquad (13)$$
$$\Rightarrow 12p_3 + 2p_2 = -8p_3 + 2q_2$$
$$\Rightarrow q_2 = 10p_3 + p_2 \qquad (14)$$
$$(8) - (10) \Rightarrow 3p_2 + 2p_1 + p_0 = 3q_2 + 2q_1 + q_0$$
$$\Rightarrow 2q_1 + q_0 = 2p_1 + p_0 - 30p_3 \qquad (15)$$
$$(15) - 2(10) \Rightarrow -54q_3 - 12q_2 + q_0 = p_0 - 6p_2 - 84p_3$$

Substituting (13) and (14) :

$$\Rightarrow 6p_3 - 120p_3 - 12p_2 + q_0 = p_0 - 6p_2 - 84p_3$$
$$\Rightarrow q_0 = 30p_3 + 6p_2 + p_0 \qquad (16)$$

From this we can deduce q_1:

$$27q_3 + 6q_2 + q_1 = 27p_3 + 6p_2 + p_1 = -3p_3 + 6(10p_3 + p_2) + q_1$$
$$\Rightarrow q_1 = p_1 - 30p_3 \qquad (17)$$

This provides us, in summary form, with four very useful equalities:

$$q_3 = -\frac{p_3}{9} \qquad (18)$$
$$q_2 = 10p_3 + p_2 \qquad (19)$$
$$q_1 = p_1 - 30p_3 \qquad (20)$$
$$q_0 = 30p_3 + 6p_2 + p_0 \qquad (21)$$

We now have a system of four equations with four variables which can be solved far more easily than six equations with eight variables. Take for example (3)≡(9) as it is rewritten:

$$\begin{aligned} Q(12) &= -192p_3 + 1440p_3 + 144p_2 + 12p_1 \\ &\quad - 360p_3 + 30p_3 + 6p_2 + p_0 \\ &= 918p_3 + 150p_2 + 12p_1 + p_0 \\ &= 12 \\ \Rightarrow 153p_3 + 25p_2 + 2p_1 + \frac{p_0}{6} &= 2 \qquad (22) \end{aligned}$$

2 Solving for $P(x)$ and $Q(x)$

By using row reduction of matricies we can find the solution to this four-equation-four-varialbe simultaneous equations problem. The form:

$$\begin{pmatrix} a_3 & a_2 & a_1 & a_0 & a \end{pmatrix} \qquad a = a_3p_3 + a_2p_2 + a_1p_1 + a_0p_0$$

is used. p_n will generally *not* represent the coefficient itself but the coefficient multiplied by the value of x within the equation. Row operations are performed in the order written thus $r_4 - r_1$ will use the r_1 which has been defined most recently, probably directly above.

$$\begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{matrix} \begin{pmatrix} 12 & 2 & 0 & 0 & 0 \\ 27 & 9 & 3 & 1 & 7 \\ 153 & 25 & 2 & \frac{1}{6} & 2 \\ 8 & 4 & 2 & 1 & 3 \end{pmatrix} \qquad \text{from} \left\{ \begin{matrix} (12) \\ (8) \\ (22) \\ (7) \end{matrix} \right.$$

$$\begin{pmatrix} 1 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & \frac{14}{9} \\ 0 & 0 & 1 & \frac{5}{42} & \frac{25}{21} \\ 0 & 0 & 0 & 1 & -\frac{89}{24} \end{pmatrix} \qquad \begin{matrix} r_1 = \frac{r_1}{12} \\ r_2 = \frac{2}{9}\left(r_2 - 27r_1\right) \\ r_3 = \frac{3}{7}\left(r_3 - 153r_1 + \frac{r_2}{2}\right) \\ r_4 = \frac{21}{8}\left(r_4 - 8r_1 - \frac{8}{3}r_2 - \frac{2}{9}r_3\right) \end{matrix}$$

$$\begin{pmatrix} 1 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & \frac{14}{9} \\ 0 & 0 & 1 & 0 & \frac{235}{144} \\ 0 & 0 & 0 & 1 & -\frac{89}{24} \end{pmatrix} \qquad \begin{matrix} r_1 = r_1 \\ r_2 = r_2 \\ r_3 = r_3 - \frac{5}{42}r_4 \\ r_4 = r_4 \end{matrix}$$

$$\begin{pmatrix} 1 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{31}{24} \\ 0 & 0 & 1 & 0 & \frac{235}{144} \\ 0 & 0 & 0 & 1 & -\frac{89}{24} \end{pmatrix} \qquad \begin{matrix} r_1 = r_1 \\ r_2 = r_2 - \frac{2}{3}r_3 - \frac{2}{9}r_4 \\ r_3 = r_3 \\ r_4 = r_4 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{31}{144} \\ 0 & 1 & 0 & 0 & \frac{31}{24} \\ 0 & 0 & 1 & 0 & \frac{235}{144} \\ 0 & 0 & 0 & 1 & -\frac{89}{24} \end{pmatrix} \qquad \begin{matrix} r_1 = r_1 - \frac{1}{6}r_2 \\ r_2 = r_2 \\ r_3 = r_3 \\ r_4 = r_4 \end{matrix}$$

Thus, from row reduction:

$$p_3 = -\frac{31}{144} \qquad (23)$$

$$p_2 = \frac{31}{24} \qquad (24)$$

$$p_1 = \frac{235}{144} \qquad (25)$$

$$p_0 = -\frac{89}{24} \qquad (26)$$

Substituting into our equations for q_i:

$$q_3 = \frac{31}{1296} \tag{27}$$
$$q_2 = -\frac{31}{36} \tag{28}$$
$$q_1 = \frac{1165}{144} \tag{29}$$
$$q_0 = -\frac{29}{12} \tag{30}$$

From these, we now have the functions defined:

$$P(x) = -\frac{31}{144}x^3 + \frac{31}{24}x^2 + \frac{235}{144}x - \frac{89}{24} \qquad x \in \mathbb{R}, 2 \leq x \leq 3 \tag{31}$$
$$Q(x) = \frac{31}{1296}x^3 - \frac{31}{36}x^2 + \frac{1165}{144}x - \frac{29}{12} \qquad x \in \mathbb{R}, 3 \leq x \leq 12 \tag{32}$$

3 Finding $R(x)$

Since $R(x)$ must be smooth where it attaches to $Q(x)$, it should satisfy the differential equation $R'(12) = Q'(12)$. Thus:

$$\begin{aligned} 3 \times 12^2 \times q_3 + 2 \times 12 \times q_2 + q_1 &= 12 \times r_1 + r_0 \\ \Rightarrow \frac{31}{9} - \frac{62}{3} + \frac{1165}{144} &= 12r_1 + r_0 \\ \Rightarrow 12r_1 + r_0 = -\frac{1315}{144} \end{aligned}$$

$R(x)$ also needs to satisfy the condition that $R(21) = 0$ so that the propeller will attach to the drive shaft at the 210cm mark as required. This then gives us $r_0 = 21$ showing that:

$$\begin{aligned} 12r_1 + 21 &= -\frac{323}{144} \\ \Rightarrow r_1 &= -\frac{3347}{1728} \\ \text{Or} & \\ R(x) &= -\frac{323}{144}x + 21 \qquad x \in \mathbb{R}, 12 \leq x \leq 21 \end{aligned} \tag{33}$$